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TITLE- Evaluation of RCS Impulse Dispersions in Attitude Hold Modes for AAP

TM- 69-1022-9

DATE- July 31, 1969

FILING CASE NO(S)- 620

AUTHOR(S)- B. D. Elrod

FILING SUBJECT(S) (ASSIGNED BY AUTHOR(S))- Attitude Control
Reaction Thrust Systems
Impulse Requirements

ABSTRACT

A model for evaluating RCS impulse dispersions in attitude hold modes due to various system parameter uncertainties is presented. The model consists of first and second order functions of the parameter uncertainties. Procedures for numerical evaluation of sensitivity coefficients in the model are described. The mean and variance of the impulse dispersion are calculated for independent, Normal parameter uncertainties.

This approach was utilized in estimating Workshop-Attitude-Control-System (WACS) impulse requirements for attitude hold of the OWS/CSM configuration in the X-POP mode during formerly planned 28 and 56 day AAP 1/2 and 2/3A missions. The results indicate that the total impulse requirement for attitude hold was 124,600 lb-sec including a 1σ impulse dispersion of 6,340 lb-sec.

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TECHNICAL MEMORANDUM

1.0 INTRODUCTION

An important planning consideration for spacecraft controlled by reaction thrust methods is the RCS propellant budget. In long duration earth orbital missions most of the propellant is expended in attitude hold operations. Estimates of propellant requirements for nominal operation in prescribed attitude modes can be achieved via computer simulation studies. However, such estimates may be deceptively low because of inevitable system and environmental parameter dispersions such as: sensor errors, gyro drift, spacecraft mass uncertainties and atmospheric density variations. Consequently, these estimates must be supplemented by a propellant or impulse dispersion* analysis, particularly if tight margins exist on propellant for other mission activities (e.g., maneuvers to support experiments).

A method is presented in this memorandum for evaluating the impulse dispersion due to various parameter uncertainties. In the next section an impulse dispersion model is described and a basis for calculating the mean and variance of the impulse dispersion is discussed. In the following section numerical results obtained from applying this approach to the Workship-Attitude-Control-System (WACS) holding the OWS/CSM configuration in the X-POP attitude mode are given.

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*Propellant dispersion is defined as the difference in the propellant requirement for attitude hold with nominal and off-nominal system and environmental parameters. The terms propellant dispersion and impulse dispersion are used synonymously. Impulse represents the force inpulse $\int F dt$ produced by a thruster firing and is related to expended propellant weight (P) by the propellant specific impulse (I_{sp}):

$$P = \int F dt / I_{sp}$$

Hence evaluation of total impulse, $\sum_i \int F_i dt$, is sufficient to determine total propellant, $\sum_i P_i$. [redacted]

(NASA-CR-106569) EVALUATION OF RCS IMPULSE DISPERSIONS IN ATTITUDE HOLD MODES FOR AAP (Bellcomm, Inc.) 23 p N79-72380

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2.0 IMPULSE DISPERSION MODEL

While a large number of parameters may be required in modeling any particular attitude control system and operating mode only certain parameters have a significant influence on the attitude hold impulse requirements. Let $(x_1, x_2, \ldots, x_n)$ represent the deviation in the significant parameters from the nominal case* and $I(x_1, x_2, \ldots, x_n)$ the functional relationship between the RCS impulse and the x_i $(i=1, \ldots, n)$. For sufficiently small parameter variations a Taylor series expansion of I about the nominal including terms only through second order yields**

$$I = I_o + \underline{b}^T \underline{x} + \frac{1}{2} \underline{x}^T A \underline{x} \qquad (1)$$

where

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad (n \times 1) \qquad (2)$$

$$\underline{b} = \begin{pmatrix} \partial I/\partial x_1 \\ \vdots \\ \partial I/\partial x_n \end{pmatrix} \Bigg|_{x_i = 0} \qquad (n \times 1) \qquad (3)$$

and

$$A = \left[\frac{\partial^2 I}{\partial x_i \partial x_j} \right] \Bigg|_{x_i, x_j = 0} \qquad (n \times n) \qquad (4)$$

Here I_o represents the nominal impulse corresponding to $\underline{x} = \underline{0}$ and the respective elements of $\underline{b}$ and A represent first and second order sensitivity coefficients. The impulse dispersion model is now defined as

$$\Delta I = (I - I_o) = \underline{b}^T \underline{x} + \frac{1}{2} \underline{x}^T A \underline{x} \qquad (5)$$

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*For the nominal case all $x_i = 0$. Non-zero mean values of x_i $(i=1, \ldots, n)$ can be accounted for by a mean impulse dispersion which will be calculated and then included in the nominal impulse required for the mode.

**T represents the matrix transpose operation.

Inclusion of the second order term in $\underline{x}$ accounts for parameter deviations which may have unidirectional* and/or cross-coupling effects on impulse dispersion. Parameter deviations which have a mutually cancelling effect** may also be included in this model, since it is only necessary to define an x_i as a linear combination of these parameters. If y_i, ..., y_{ir} represent such parameters, then

$$x_i = \sum_k^r d_k y_{ik} \tag{6}$$

where the d_k are known constants associated with a particular attitude control system model.

Use of the model in Eq. (5) for calculation of impulse dispersions requires evaluation of the sensitivity coefficients in $\underline{b}$ and A corresponding to the parameter deviations $\underline{x}$. Evaluation of sensitivity coefficients is treated in the following subsections.

Parameter deviations generally are not known exactly but only in some probabilistic sense. Evaluation of the mean and variance of the impulse dispersion assuming that the elements of $\underline{x}$ are independent, Normal random variables is treated in Section 2.2. If the x_i are not Normal and probability distributions are available, the model in Eq. (5) can be used for a Monte Carlo analysis to determine the distribution function and other statistical parameters of ΔI. This approach utilizes numerous calculations of ΔI with $\underline{x}$ obtained by random sampling of the x_i from individual distributions.

2.1 Numerical Evaluation of Sensitivity Coefficients

An analytical relationship for the RCS impulse I is virtually impossible to obtain in practice. Consequently numerical evaluation of the sensitivity coefficients is necessary. This involves computer simulation of the attitude control system for a particular attitude mode with various values of x_i chosen from the expected range of variation.

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*For instance gyro drift alone with spacecraft held in the POP attitude mode increases impulse requirements regardless of the algebraic sign associated with the drift.

**Examples are attitude sensor error and the error in location of spacecraft principal axes due to uncertainties in mass characteristics.

The RCS impulse obtained with all $x_i = 0$ is the nominal, I_o. Impulse differences $[I(x_i)-I_o]$ obtained for various values of a particular parameter deviation (x_i) may be used to plot an impulse dispersion characteristic such as shown in Figure 1.

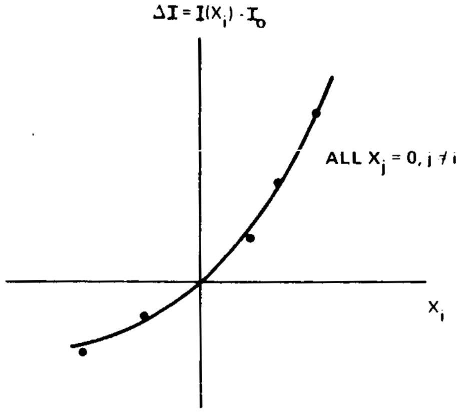

FIGURE 1 - IMPULSE DISPERSION CHARACTERISTIC FOR PARAMETER X_i

The respective b_i and a_{ii} elements in $\underline{b}$ and A can then be obtained by a linear + parabolic fit to this characteristic, namely

$$\Delta I(x_i) = b_i x_i + \frac{1}{2} a_{ii} x_i^2 \tag{7}$$

Impulse differences obtained for various values of two parameters $(x_i, x_j \neq 0)$ yield a family of impulse dispersion characteristics such as shown in Figure 2a. The slope $\partial I/\partial x_i$ at $x_i = 0$ for each curve (x_j = constant) yields the dispersion slope characteristic shown in Figure 2b. The slope of a straight line fit to this data represents the element a_{ij} in A.

The amount of data required to evaluate sensitivity coefficients in this manner increases rapidly with the number of parameters (n). Due to computer time constraints RCS impulse $I(x_i)$ would be evaluated, as a practical matter, on a per orbit

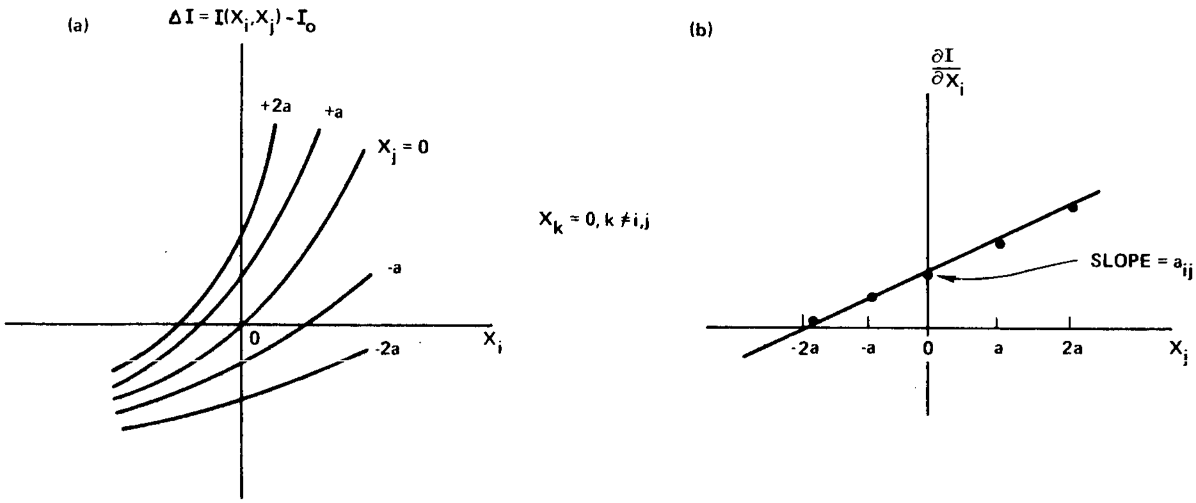

FIGURE 2 - IMPULSE DISPERSION AND DISPERSION SLOPE CHARACTERISTICS FOR PARAMETERS X_i AND X_j

basis rather than over a whole mission.* Thus, if I_{ok} and ΔI_k represent the nominal and dispersion impulse for the kth orbit, the total impulse in an N orbit mission is

$$I_T = \sum_{k=1}^{N} (I_{ok} + \Delta I_k) \tag{8}$$

However, if long term effects such as altitude decay and β angle changes have an insignificant effect on the sensitivity coefficients the total impulse is

$$I_T = \sum_{k=1}^{N} I_{ok} + N\Delta I \tag{9}$$

where ΔI represents the dispersion impulse per orbit based on Eq. (5) with $\underline{b}$ and A constant. Depending on mission duration and orbital paramters the nominal impulse I_{ok} also may not vary

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*Effects of arbitrary initial conditions (spacecraft attitude and rate) generally can be accounted for by computing the average impulse/orbit in a 10-20 orbit simulation.

appreciably. In that case a reasonable approximation is to average the best and worst case nominal impulse/orbit. Consequently, I_T becomes

$$I_T \approx N\left(\frac{I_{ob} + I_{ow}}{2} + \Delta I\right) \tag{10}$$

where I_{ob} and I_{ow} represent impulse/orbit for best and worst case conditions on slowly varying parameters (e.g., orbital altitude and β angle).

2.2 Mean and Variance of Impulse Dispersion

In the absence of hard statistical data on all parameter deviations one approach to determining the impact on impulse dispersions is to treat all x_i as independent, Normal random variables with mean m_i and variance σ_i^2. The corresponding mean and variance of the impulse dispersion are given by

$$\overline{\Delta I} \equiv E(\Delta I) = E[\underline{b}^T\underline{x} + \frac{1}{2}\,\underline{x}^T A\,\underline{x}] \tag{11}$$

and

$$\sigma^2(\Delta I) = E(\Delta I^2) - \overline{\Delta I}^2 = E[(\underline{b}^T\underline{x} + \frac{1}{2}\,\underline{x}^T A\,\underline{x})^2] - \overline{\Delta I}^2 \tag{12}$$

Expressions for $\overline{\Delta I}$ and $\sigma^2(\Delta I)$ are derived in Appendix A. The results are

$$\overline{\Delta I} = \underline{b}^T\underline{m} + \frac{1}{2}\,(\sigma^T A\underline{\sigma} + \underline{m}^T A\,\underline{m}) \tag{13}$$

and

$$\sigma^2(\Delta I) = \frac{1}{2}\hat{\underline{\sigma}}^T\bar{A}\hat{\underline{\sigma}} + (\underline{b}+A\,\underline{m})^T\,\Lambda\,(\underline{b}+A\,\underline{m}) \tag{14}$$

where

$$\underline{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}, \quad \underline{\sigma} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}, \quad \underline{\hat{\sigma}} = \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_n^2 \end{pmatrix} \quad (n \times 1) \tag{15}$$

$$\Lambda = \begin{bmatrix} \sigma_1^2 & & O \\ & \ddots & \\ O & & \sigma_n^2 \end{bmatrix} \quad (n \times n) \tag{16}$$

$$A = \begin{bmatrix} a_{11} & & O \\ & \ddots & \\ O & & a_{nn} \end{bmatrix} \quad (n \times n) \tag{17}$$

and

$$\bar{A} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{12}^2 & a_{22}^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1n}^2 & \cdots & \cdots & a_{nn}^2 \end{bmatrix} \quad (n \times n) \tag{18}$$

Two special cases are of interest: a) no cross-product terms ($a_{ij} = 0$) and b) all x_i have zero means ($m_i = 0$).

a) $\underline{a_{ij} = 0}$: $A = A$, $\bar{A} = A^2$

$$\overline{\Delta I} = \underline{b}^T \underline{m} + \frac{1}{2} (\underline{\sigma}^T A \, \underline{\sigma} + \underline{m}^T A \, \underline{m}) \tag{19}$$

$$\sigma^2 (\Delta I) = \frac{1}{2} \underline{\hat{\sigma}}^T A^2 \underline{\hat{\sigma}} + (\underline{b} + A \, \underline{m})^T \, \Lambda \, (\underline{b} + A \, \underline{m}) \tag{20}$$

b) $\underline{m_i = 0}$:

$$\overline{\Delta I} = \frac{1}{2}\,\underline{\sigma}^T A\,\underline{\sigma} \tag{21}$$

$$\sigma^2(\Delta I) = \underline{b}^T \Lambda \underline{b} + \frac{1}{2}\hat{\underline{\sigma}}^T \bar{\Lambda} \hat{\underline{\sigma}} \tag{22}$$

In the second case it is observed that $\overline{\Delta I} \neq 0$, as a result of the second order term in the impulse dispersion model, Eq. (5).

3.0 APPLICATION TO WACS IMPULSE REQUIREMENT

In AAP missions 1/2 and 2/3A attitude hold of the OWS/CSM configuration was to be performed by the WACS primarily in the X-POP mode* for 28 and 56 day periods. In this section numerical results obtained in applying the foregoing approach to estimating WACS impulse requirements for X-POP attitude hold are presented. A digital simulation of the WACS was utilized to obtain all impulse data. A simplified block diagram is shown in Figure 3.

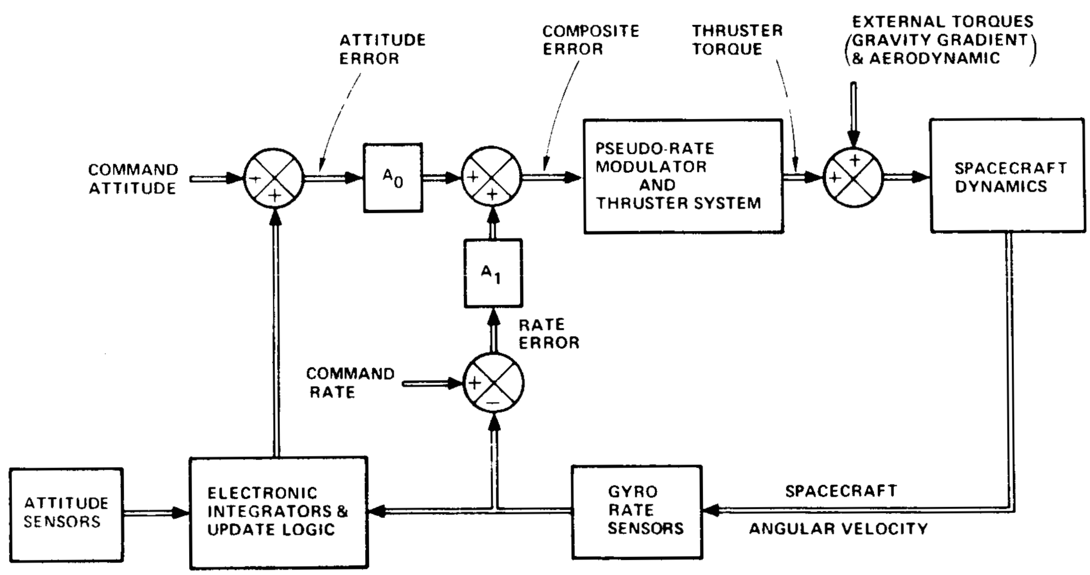

FIGURE 3 - SIMPLIFIED BLOCK DIAGRAM OF WACS

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*In the X-POP mode the spacecraft roll (x) axis is nominally aligned perpendicular to the orbital plane and the pitch (y) axis aligned normal to the sun line.[1] See Figure C after Appendix B. Control requirements include ±0.5° attitude hold on all axes with attitude updating once per orbit via horizon sensors (pitch, yaw) and a sun sensor (roll).[1]

While many parameters have an effect on impulse requirements, those considered to have the most impact are:*

1. drift in gyro rate sensors and electronic integrators (pitch, yaw axes);

2. inaccuracy of horizon sensors (pitch, yaw axes);

3. uncertainty in difference between pitch and yaw axes inertias;

4. uncertainty in CM location (x axis component);

5. uncertainty in atmospheric density.

In Table 1 the parameters are listed along with corresponding sensitivity coefficients and statistical data (m_i, σ_i) used for each x_i.

In the course of evaluating the sensitivity coefficients** it was determined that a simple parabolic fit (b_i=0, $a_{ii} \neq 0$) could be made in four cases, a simple linear fit ($b_i \neq 0$, $a_{ii} = 0$) could be made in the other three cases, and in nearly all cases cross-product coefficients (a_{ij}) were zero or had negligible effect. Furthermore, long term effects (altitude decay-10 NM, β change-58.5°) had only slight effect on the nominal impulse as indicated by I_{ob} and I_{ow} in Table 2. Hence, the simpler result in Eq. (10) is applicable for evaluating total impulse I_T. Since all $a_{ij} \neq 0$, the expressions in Eqs. (13) and (14) are used to evaluate $\overline{\Delta I}$ and $\sigma^2(\Delta I)$.

The mean total impulse $\overline{I}_T$ is then given by

$$\overline{I}_T = N\left(\frac{I_{ob} + I_{ow}}{2} + \overline{\Delta I}\right) = I_{oT} + N\overline{\Delta I} \qquad (24)$$

where I_{oT} represents the nominal impulse over N orbits.

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*Other factors which significantly affect impulse requirements are thruster impulse level and system error deadbands. However, these were considered fixed at the design values stated in Reference 1. Sun sensor inaccuracy and gyro/integrator drift associated with roll axis control had negligible effect on impulse requirements.

**Impulse characteristics corresponding to Figures 1 and 2 are included in Appendix B as well as other reference information on data used in the WACS simulation.

TABLE 1. PARAMETERS AND DATA FOR WACS IMPULSE DISPERSION MODEL

Parameter X_i	Definition	Statistical parameters* m_i	σ_i	b_{ii}	Sensitivity Coefficients a_{ii}	a_{ij}
X_1	GRSEI Drift** (pitch axis)	0.1 °/hr	0.3 °/hr	$b_1 = 0$	$a_{11} = 17.6 \frac{\text{lb-sec}}{(\text{deg/hr})^2}$	$a_{13} = 9.8 \frac{\text{lb-sec}}{(\text{deg}^2/\text{hr})}$
X_2	GRSEI Drift** (yaw axis)	0.1 °/hr	0.3 °/hr	$b_2 = 0$	$a_{22} = 16.1 \frac{\text{lb-sec}}{(\text{deg/hr})^2}$	$a_{24} = 6.95 \frac{\text{lb-sec}}{(\text{deg}^2/\text{hr})}$
X_3	Horizon Sensor Inaccuracy (pitch axis)	0	0.5 °	$b_3 = 0$	$a_{33} = 13.8 \frac{\text{lb-sec}}{(\text{deg})^2}$	
X_4	Horizon Sensor Inaccuracy (yaw axis)	0	0.5 °	$b_4 = 0$	$a_{44} = 3.5 \frac{\text{lb-sec}}{(\text{deg})^2}$	all other $a_{ij} = 0$
X_5	CM Location Uncertainty (roll axis)	0	5 in.	$b_5 = -0.06 \frac{\text{lb-sec}}{\text{in.}}$	$a_{55} = 0$	
X_6	Pitch/Yaw Axis Inertia Difference Uncertainty	0	4×10^3 slug-ft^2	$b_6 = 0.61 \times 10^{-3} \frac{\text{lb-sec}}{\text{slug-ft}^2}$	$a_{66} = 0$	
X_7	Atmospheric Density Uncertainty	0	1.4×10^{-15} slugs/ft^3	$b_7 = 0.40 \times 10^{15} \frac{\text{lb-sec}}{\text{slugs/ft}^3}$	$a_{77} = 0$	

*Data is based on treating estimated limits on parameter variations as $(m_i + 3\sigma_i)$ (e.g., Maximum GRSEI Drift is 1°/hr). Sources for estimated limits on parameter variations: a) x_1, x_2, x_3, x_4 – Ref. 5, 6 ,b) x_5, x_6–Ref. 6, c) x_7 – Ref. 7 (2σ data given, pp. 19, 45).

**GRSEI Gyro Rate Sensor and Electronic Integrator
(Although drift due to gyro rate sensors and electronic integrators affect rate and attitude independently, their net effect can be treated on each axis as a single drift. See Section 2.0 and Eq. 6).

The dispersion about this mean impulse is $\sigma(I_T) = N\sigma(\Delta I)$.

The values of I_{OT}, $N\overline{\Delta I}$ and $\sigma(I_T)$ corresponding to the data in Table 1 are tabulated in Table 2 for both 28 and 56 day missions. The relative importance of the various parameter deviations is shown in Table 3 which lists $\overline{\Delta I}$ and $\sigma(\Delta I)$ for each x_i taken separately, and in pairs (x_i, x_j) when cross-coupling is not negligible $(a_{ij} \neq 0)$. On the basis of this analysis the mean impulse requirement, $\overline{I}_T$, for X-POP attitude hold of the OWS/CSM configuration for 28 and 56 day durations is 118,260 lb-sec with a standard deviation $\sigma(I_T) = 6,340$ lb-sec. Basing impulse requirements on a 1σ upper limit, namely $[\overline{I}_T + \sigma(I_T)]$, yields 124,600 lbs-sec as the requirement.

TABLE 2 - IMPULSE DATA

IMPULSE/ORBIT (LB-SEC)					IMPULSE/MISSION (LB-SEC)					
NOMINAL			DISPERSION		28 DAYS ***			56 DAYS****		
I_{ow} **	I_{ob} *	$\frac{I_{ob} + I_{ow}}{2}$	$\overline{\Delta I}$	$\sigma(\Delta I)$	I_{oT}	$N\overline{\Delta I}$	$\sigma(I_T)$	I_{oT}	$N\overline{\Delta I}$	$\sigma(I_T)$
89.6	82.6	86.1	4.0	4.8	37,550	1,750	2,100	75,445	3,515	4,240

* ORBITAL ALTITUDE-210 NM; β=58.5°

** ORBITAL ALTITUDE-200 NM; β=0°

*** N = 436 ORBITS (210 NM AVERAGE ORBIT ALTITUDE-28 DAYS)

**** N = 876 ORBITS (200 NM AVERAGE ORBIT ALTITUDE-56 DAYS)

TABLE 3 - RELATIVE EFFECT OF PARAMETER DEVIATIONS ON IMPULSE DISPERSION

DISPERSION (LB-SEC/ORBIT) \ X_i	X_1	X_3	X_1, X_3	X_2	X_4	X_2, X_4	X_5	X_6	X_7
$\overline{\Delta I}$	0.89	1.76	2.65	0.85	0.45	1.30	0	0	0
$\sigma(\Delta I)$	1.27	2.49	3.16	1.21	0.71	1.76	0.3	2.44	0.56

4.0 SUMMARY AND CONCLUSIONS

Evaluation of RCS impulse dispersion (ΔI) due to various parameter uncertainties ($x_1, \ldots, x_n$) in attitude hold modes has been formulated in terms of an impulse dispersion model. The model includes first and second order functions of the parameter uncertainties, namely*

$$\Delta I = \underline{b}^T \underline{x} + \frac{1}{2} \underline{x}^T A \, \underline{x} \quad , \quad \underline{x}^T = (x_1 \ldots, x_n) \tag{25}$$

where $\underline{b}$ and A are (n x 1) and (n x n) sensitivity coefficient matrices. Numerical evaluation of sensitivity coefficients based on impulse data obtained from computer simulation of the attitude control system has been described. The results of evaluating the mean impulse dispersion ($\overline{\Delta I}$) and the variance of the impulse dispersion $\sigma^2(\Delta I)$ under the assumption of independent, Normal parameter uncertainties are given in Eqs. (13) and (14).

WACS impulse requirements for attitude hold of the OWS/CSM configuration in the X-POP mode during 28 and 56 day AAP 1/2 and 2/3A missions were evaluated on the basis of this model. Parameter uncertainties considered were horizon sensor inaccuracy, gyro and electronic integrator drift as well as uncertainties in atmospheric density and spacecraft mass characteristics. The total impulse requirement for X-POP attitude hold in the 28 day and 56 day missions was found to be 124,600 lb-sec including a 1σ impulse dispersion of 6,340 lb-sec.

Acknowledgement

In developing the WACSIM computer program the author is grateful for the steadfast support of Mrs. Nancy Kirkendall.

1022-BDE-mef

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*T represents the matrix transpose operation.

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APPENDIX A

Evaluation of $\overline{\Delta I}$ and $\sigma^2(\Delta I)$

The purpose of this appendix is to outline the procedure for calculating the mean and variance of the function

$$\Delta I = \underline{b}^T \underline{x} + \frac{1}{2} \underline{x}^T A \, \underline{x} \qquad \text{(A-1)}$$

where $\underline{b}$ is (n x 1) and A is (n x n) and symmetric. The elements of $\underline{x}$ are independent, jointly Normal random variables so that

$$E(x_i) = m_i \qquad i = 1, \ldots, n \qquad \text{(A-2)}$$

$$E(x_i x_j) = \begin{cases} \sigma_i^2 + m_i^2 & i=j \\ m_i \, m_j & i \neq j \end{cases} \qquad \text{(A-3)}$$

$$E(x_i x_j x_k) = \begin{cases} m_i(3\sigma_i^2 + m_i^2) & i=j=k \\ m_k(\sigma_i^2 + m_i^2) & i=j\neq k \\ m_i \, m_j \, m_k & i\neq j\neq k \end{cases} \qquad \text{(A-4)}$$

$$E(x_i x_j x_k x_l) = \begin{cases} (3\sigma_i^4 + 6\sigma_i^2 m_i^2 + m_i^4) & i=j=k=l \\ m_l \, m_i \, (3\sigma_i^2 + m_i^2) & i=j=k\neq l \\ (\sigma_i^2 + m_i^2)\,(\sigma_k^2 + m_k^2) & i=j\neq k=l \\ m_k \, m_l \, (\sigma_i^2 + m_i^2) & i=j\neq k\neq l \\ m_i \, m_j \, m_k \, m_l & i\neq j\neq k\neq l \end{cases} \qquad \text{(A-5)}$$

If the elements of $\underline{x}$ are independent, but not Normal, moments of the x_i up to fourth order must be known.

APPENDIX A

In subsequent calculations the following vectors and matrices are used

$$\underline{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}, \quad \underline{\sigma} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}, \quad \hat{\underline{\sigma}} = \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_n^2 \end{pmatrix} \qquad (n \times 1) \qquad \text{(A-6)}$$

$$\Lambda = \begin{bmatrix} \sigma_1^2 & & \bigcirc \\ & \ddots & \\ \bigcirc & & \sigma_n^2 \end{bmatrix} \qquad (n \times n) \qquad \text{(A-7)}$$

$$\bar{A} = \begin{bmatrix} a_{11}^2 & a_{12} & \cdots & a_{1n}^2 \\ a_{12}^2 & a_{22}^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1n}^2 & \cdots & \cdots & a_{nn}^2 \end{bmatrix} \qquad (n \times n) \qquad \text{(A-8)}$$

$$A = \begin{bmatrix} a_{11} & & \bigcirc \\ & \ddots & \\ \bigcirc & & a_{nn} \end{bmatrix} \qquad (n \times n) \qquad \text{(A-9)}$$

<u>A.1 Mean Value of ΔI:</u> $\overline{\Delta I}$

$$\overline{\Delta I} = E(\Delta I) = \sum_i^n b_i E(x_i) + \frac{1}{2} \sum_i^n \sum_j^n a_{ij} E(x_i \; x_j) \qquad \text{(A-10)}$$

APPENDIX A

Substitution of Eqs. (A-2) and A-3) into (A-10) and arrangement of terms yields

$$\overline{\Delta I} = \underline{b}^T \underline{m} + \frac{1}{2} (\underline{\sigma}^T A \ \underline{\sigma} + \underline{m}^T A \ \underline{m}) \tag{A-11}$$

A.2 Variance of ΔI: $\sigma^2(\Delta I)$

$$\begin{aligned} \sigma^2(\Delta I) &= E(\Delta I^2) - \overline{\Delta I}^2 \\ &= E[(\underline{b}^T \underline{x})^2] + E[(\underline{b}^T \underline{x})(\underline{x}^T A \ \underline{x})] + \frac{1}{4} E[(\underline{x}^T A \ \underline{x})^2] - \overline{\Delta I}^2 \end{aligned} \tag{A-12}$$

A.2.1 $E[(\underline{b}^T \underline{x})^2]$

$$\begin{aligned} E[(\underline{b}^T \underline{x})^2] &= \sum_i^n \sum_j^n b_i \ b_j \ E(x_i x_j) \\ &= \underline{b}^T \ \Lambda \ \underline{b} + (\underline{b}^T \underline{m})^2 \end{aligned} \tag{A-13}$$

A.2.2 $E[(\underline{b}^T \underline{x})(\underline{x}^T A \ \underline{x})]$

$$\begin{aligned} E[(\underline{b}^T \underline{x})(\underline{x}^T A \ \underline{x})] &= E\left[\sum_k^n b_k \ x_k \sum_i^n \sum_j^n a_{ij} \ x_i \ x_j \right] \\ &= \sum_i^n b_i \ a_{ii} \ E(x_i^{\,3}) + \sum_{\substack{k \\ k \neq i}}^n \sum_i^n b_k \ a_{ii} \ E(x_i^{\,2} x_k) \\ &\quad + 2\sum_{\substack{k \\ k\neq i}}^n \sum_i^n b_k \ a_{ik} E(x_i x_k^{\,2}) + \sum_{\substack{i \\ i\neq j\neq k}}^n \sum_j^n \sum_k^n b_k \ a_{ij} E(x_i x_j x_k) \end{aligned} \tag{A-14}$$

APPENDIX A

Substitution of Eq. (A-4) into (A-14) and arrangement of terms yields

$$E[(\underline{b}^T\underline{x})(\underline{x}^T A\ \underline{x})] = (\underline{b}^T\underline{m})(\underline{m}^T A\ \underline{m}) + (\underline{b}^T\underline{m})(\underline{\sigma}^T A\ \underline{\sigma}) + 2\ \underline{b}^T \Lambda A\ \underline{m} \tag{A-15}$$

A.2.3 $E[(\underline{x}^T A\ \underline{x})^2]$

$$E[(\underline{x}^T A\ \underline{x})^2] = \sum_i^n \sum_j^n \sum_k^n \sum_l^n a_{ij} a_{kl} E(x_i x_j x_k x_l)$$

$$= \sum_i^n a_{ii}{}^2 E(x_i{}^4) + \sum_{\substack{i \\ i\neq k}}^n \sum_k^n a_{ii} a_{kk} E(x_i{}^2 x_k{}^2) + 2 \sum_{\substack{i \\ i\neq j}}^n \sum_j^n a_{ij}{}^2 E(x_i{}^2 x_j{}^2)$$

$$+ 4 \sum_{\substack{i \\ i\neq j}}^n \sum_j^n a_{ij} a_{jj} E(x_i x_j{}^3) + 4 \sum_{\substack{i \\ i\neq j\neq k}}^n \sum_j^n \sum_k^n a_{ij} a_{kk} E(x_i x_j x_k{}^2)$$

$$+ 4 \sum_{\substack{i \\ i\neq j\neq k}}^n \sum_j^n \sum_k^n a_{ij} a_{jk} E(x_i x_j{}^2 x_k) + \sum_{\substack{i \\ i\neq j\neq k\neq l}}^n \sum_j^n \sum_k^n \sum_l^n a_{ij} a_{kl} E(x_i x_j x_k x_l) \tag{A-16}$$

Substitution of Eq. (A-5) into (A-16) and arrangement of terms yields

$$E[(\underline{x}^T A\ \underline{x})^2] = (\underline{m}^T A\ \underline{m})^2 + (\underline{\sigma}^T A\ \underline{\sigma})^2 + 2\underline{\hat{\sigma}}^T \bar{A}\ \underline{\hat{\sigma}} + 2(\underline{\sigma}^T A \underline{\sigma})(\underline{m}^T A\ \underline{m})$$

$$+ 4\ \underline{m}^T A \Lambda A\ \underline{m} \tag{A-17}$$

A.2.4 Calculation of $\sigma^2(\Delta I)$

Substitution of Eqs. (A-11), (A-13), (A-15) and (A-17) into (A-12 yields

$$\sigma^2(\Delta I) = \frac{1}{2}\hat{\underline{\sigma}}^T \bar{A}\ \hat{\underline{\sigma}} + \underline{b}^T \Lambda\ \underline{b} + 2\underline{b}^T \Lambda A \underline{m} + \underline{m}^T A \Lambda A\ \underline{m}$$

$$= \frac{1}{2}\hat{\underline{\sigma}}^T \bar{A}\ \hat{\underline{\sigma}} + (\underline{b} + A\ \underline{m})^T\ \Lambda\ (\underline{b} + A\ \underline{m}) \qquad \text{(A-18)}$$

APPENDIX B

WACS Simulation Parameters and Impulse Dispersion Characteristics

This appendix contains pertinent WACS impulse dispersion characteristics and reference information on input data used in the WACS simulation.

B.1 Impulse Dispersion Characteristics

In Figures B.1 and B.2 are shown impulse dispersion characteristics corresponding to Figure 1 for parameters $(x_1,...,x_7)$ defined in Table 1. Impulse dispersion and dispersion slope characteristics corresponding to Figure 2 are shown in Figure B-3 for parameters (x_1,x_3) and (x_2,x_4). Dots on each graph denote data points obtained from the WACS simulation.*

B.2 WACS Simulation Parameters

a. Mass Characteristics - inertias, center of mass location and principal axes direction cosines based on data in Reference 2 (Case 7).

b. Control Law Parameters - based on data in Reference 1.

c. Atmospheric Density - based on MSFC atmospheric density model described in Reference 3.

d. Aerodynamic Torque Model - analytical model based on cylinder and flat plate approximation to OWS/CSM configuration.**

e. Orbital Parameters - inclination = 35°

orbital altitude = 210 NM

f. Thruster Minimum Impulse - 1.25 lb-sec.

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* Data points on each graph represent average impulse/orbit based on 10-20 orbit simulations.

** This analytical model is in close agreement with MSFC aerodynamic data in Reference 4. Details of the model and a comparison with MSFC aerodynamic data will be reported in a future memorandum.

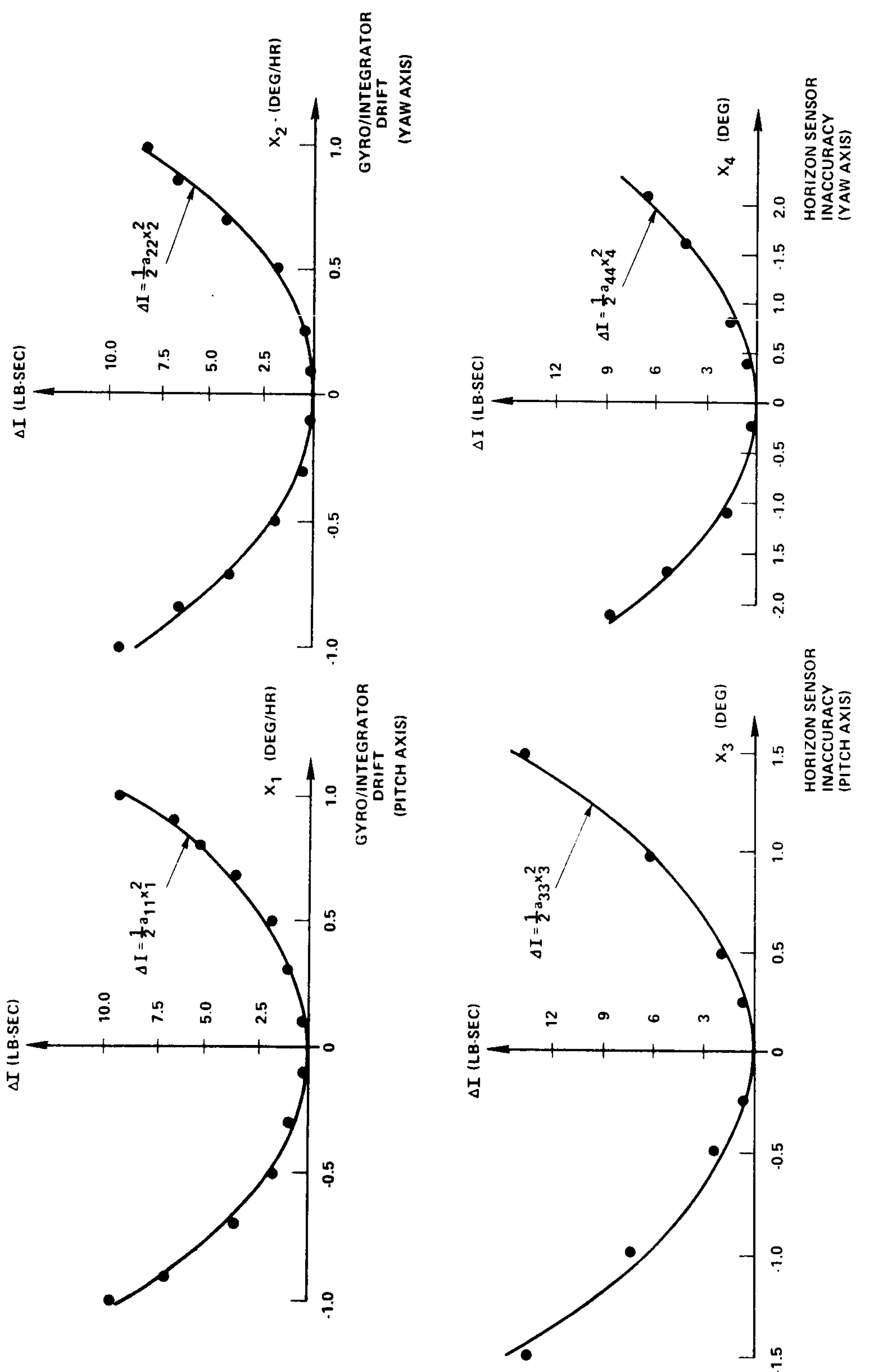

FIGURE B-1 - IMPULSE DISPERSION CHARACTERISTICS (ΔI vs X_1, X_2, X_3, X_4)

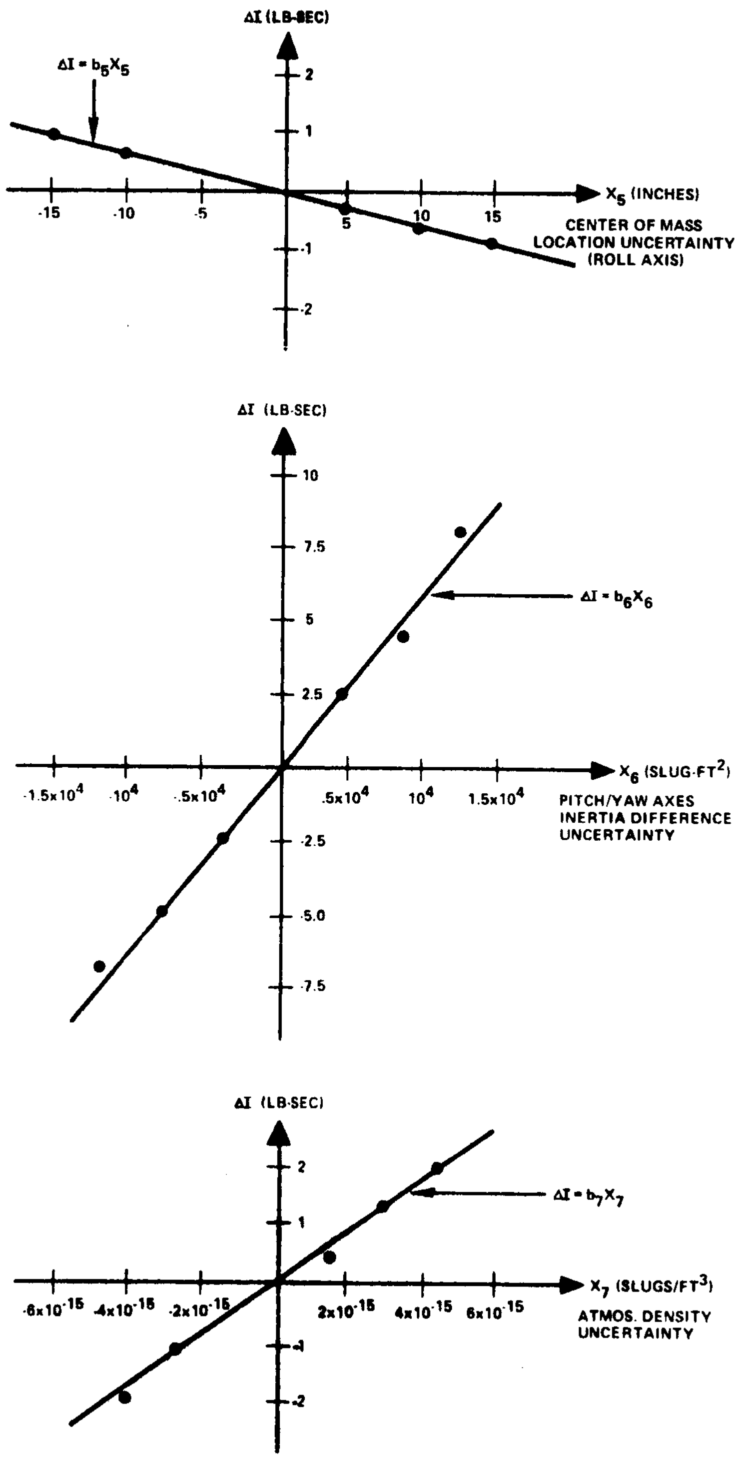

FIGURE B-2 - IMPULSE DISPERSION CHARACTERISTICS (ΔI vs X_5, X_6, X_7)

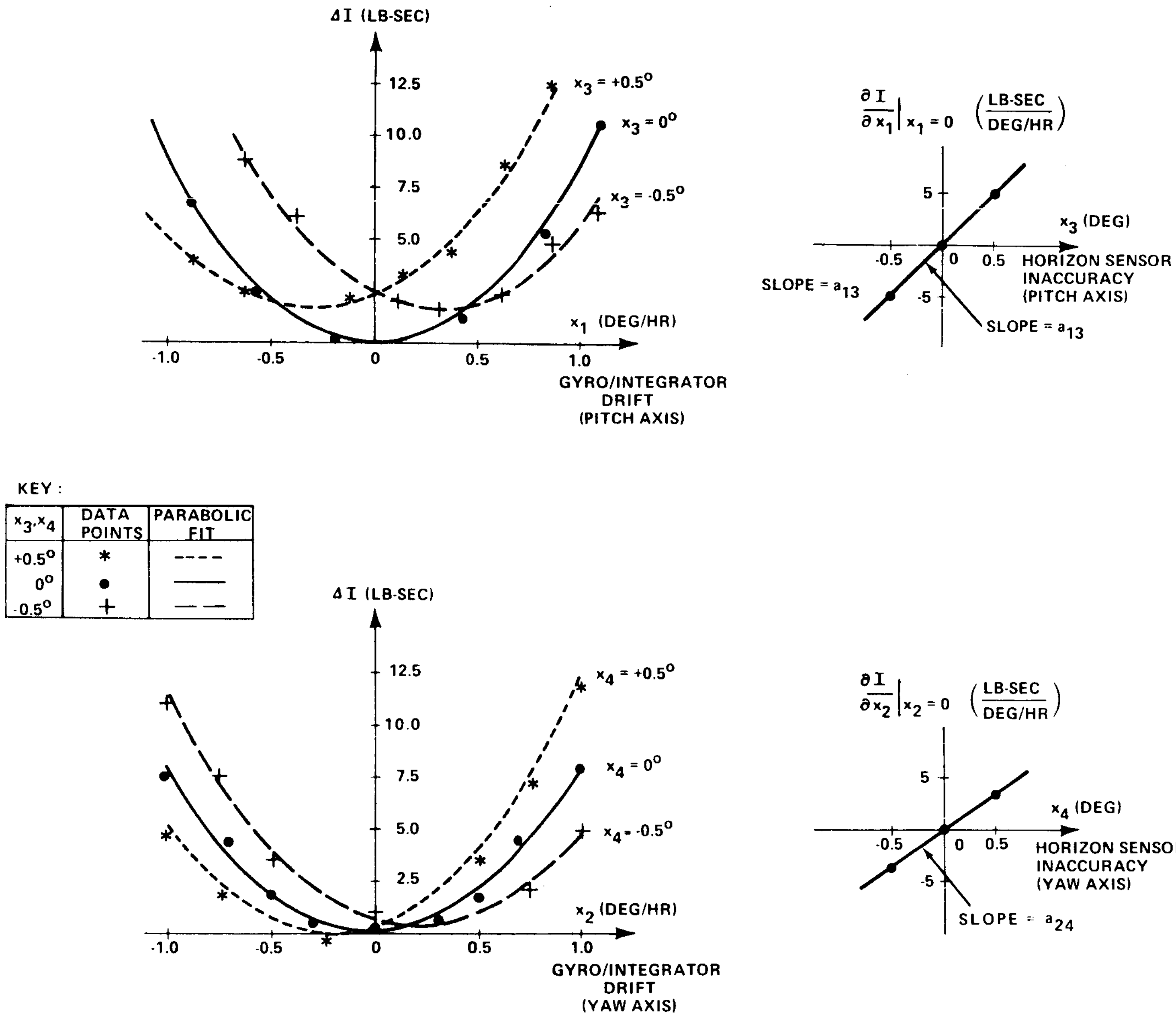

FIGURE (B-3) - IMPULSE DISPERSION AND DISPERSION SLOPE CHARACTERISTICS FOR PARAMETER DEVIATIONS (x_1,x_3) AND (x_2,x_4)

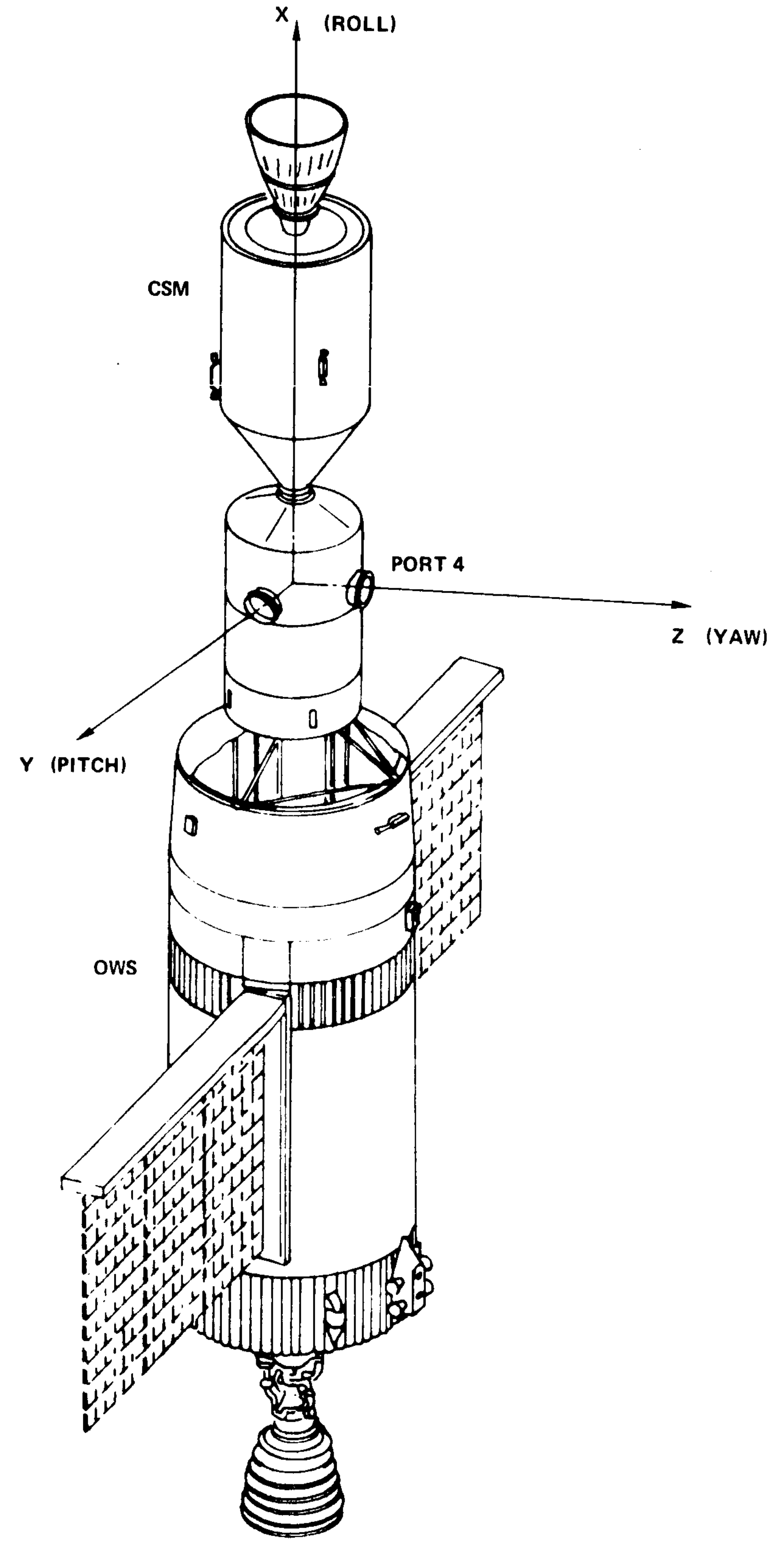

FIGURE C - OWS/CSM CONFIGURATION